

Does Cosmic No-Hair Conjecture in Brane Scenarios follow from General Relativity?

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In this paper we examine the Cosmic No-Hair Conjecture (CNHC) in brane world scenarios. For the validity of this conjecture, in addition to the strong and weak energy conditions for the matter field, a similar type of assumption is to be made on the quadratic correction term and there is a restriction on the non-local term. It is shown by examples with realistic fluid models that strong and weak energy conditions are sufficient for CNHC in brane world.

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I. INTRODUCTION

The idea of brane world scenarios may resolve the challenging problem in theoretical physics namely the unification of all forces and particles in nature. It is suggested that we live in a four dimensional brane embedded in a higher dimensional space-time. As a result, the fundamental higher dimensional Planck mass could be of same order as the electro weak scale and thereby one of the hierarchy problems in the current standard model of high-energy physics are resolved [1-4].

According to Randall and Sundrum [3, 4] it is possible to have a single massless bound state confined to a domain wall or 3-brane in five dimensional non-factorizable geometries. They have shown that this bound state corresponds to the zero mode of the Kaluza-Klein dimensional reduction and is related to the four dimensional gravitation [2]. Hence all matter and gauge fields (except gravity) are confined to a 3-brane embedded in a five dimensional space-time (bulk) while gravity can propagate in the bulk. As a consequence, the gravity on the brane can be described by the Einstein's equations modified by two additional terms, namely (i) quadratic in matter variables and (ii) the electric part of the five dimensional Weyl tensor [5].

In general terminology, the CNHC [6, 7] states that “all expanding universe models with a positive cosmological constant asymptotically approach the de-Sitter solution”. To address the question whether the universe evolves to a homogeneous and isotropic state during an inflationary epoch, Gibbons and Hawking [6] and then Hawking and Moss [7] developed this conjecture. Subsequently, Wald [8] gave a formal proof of it for homogeneous cosmological models (Bianchi models) with a positive cosmological constant. He assumed that the matter field should satisfy strong and weak energy conditions.

In this paper we wish to extend Wald's [8] result in the brane world scenario and examine whether the new conditions can be minimize using those in general relativity.

II. COSMIC NO HAIR CONJECTURE IN BRANE WORLD

According to Roy Maartens [1], the Einstein equations on the brane can be written as

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa_4^2 T_{\mu\nu} + \kappa_5^4 S_{\mu\nu} - E_{\mu\nu} \quad (1)$$

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where $S_{\mu\nu}$ and $T_{\mu\nu}$ are the two correction terms (local and non-local) in the energy momentum tensor. The local correction term $S_{\mu\nu}$ has the expression

$$4S_{\mu\nu} = \frac{1}{3}TT_{\mu\nu} - T_{\mu\rho}T_{\nu}^{\rho} - \frac{1}{2}g_{\mu\nu}\left(\frac{1}{3}T^2 - T_{\rho\sigma}T^{\rho\sigma}\right) \quad (2)$$

while $E_{\mu\nu}$ is the electric part of the 5D Weyl tensor in the bulk. Now the scalar constraint (initial value constraint) equation and the Roychoudhuri equation on the brane has the form

$$G_{\mu\nu}n^{\mu}n^{\nu} = \Lambda + \kappa_4^2 T_{\mu\nu}n^{\mu}n^{\nu} + \kappa_5^4 S_{\mu\nu}n^{\mu}n^{\nu} - E_{\mu\nu}n^{\mu}n^{\nu} \quad (3)$$

and

$$R_{\mu\nu}n^{\mu}n^{\nu} = -\Lambda + \kappa_4^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)n^{\mu}n^{\nu} + \kappa_5^4\left(S_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S\right)n^{\mu}n^{\nu} - E_{\mu\nu}n^{\mu}n^{\nu} \quad (4)$$

where n^{μ} is the unit normal to the spatial homogeneous hypersurfaces. In terms of the homogeneous hypersurface elements namely, the projected metric $h_{\mu\nu}(=g_{\mu\nu} + n_{\mu}n_{\nu})$ and the extrinsic curvature $K_{\mu\nu}(= \nabla_{\nu}n_{\mu})$ and using the Gauss-Codazzi equations the above two equations namely equations (3) and (4) become

$$K^2 = 3\Lambda + \frac{3}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{3}{2}{}^{(3)}R + 3\kappa_4^2 T_{\mu\nu}n^{\mu}n^{\nu} + 3\kappa_5^4 S_{\mu\nu}n^{\mu}n^{\nu} - 3E_{\mu\nu}n^{\mu}n^{\nu} \quad (5)$$

and

$$\dot{K} = \Lambda - \frac{1}{3}K^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} - \kappa_4^2\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)n^{\mu}n^{\nu} - \kappa_5^4\left(S_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S\right)n^{\mu}n^{\nu} + E_{\mu\nu}n^{\mu}n^{\nu} \quad (6)$$

where the dot denotes the Lie derivative with respect to proper time, K is the trace of the extrinsic curvature, $\sigma_{\mu\nu}$ is the shear of the time like geodesic congruence orthogonal to the homogeneous hypersurfaces and ${}^{(3)}R$ is the scalar curvature of the homogeneous hypersurfaces.

Using the idea of Wald and proceeding along his approach (for details see Wald et al [8] and Chakraborty et al [9]) one can find that for CNHC we must have

$$(a) \quad S_{\mu\nu}n^{\mu}n^{\nu} \geq 0 \quad \text{and} \quad \left(S_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S\right)n^{\mu}n^{\nu} \geq 0 \quad (7)$$

and

$$(b) \quad E_{\mu\nu}n^{\mu}n^{\nu} \leq 0 \quad (8)$$

in addition to the weak and strong energy conditions for the matter field

$$T_{\mu\nu}n^{\mu}n^{\nu} \geq 0 \quad \text{and} \quad \left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right)n^{\mu}n^{\nu} \geq 0 \quad (9)$$

Now if we use the expression (2) for $S_{\mu\nu}$ in (7) then we get

$$\frac{1}{3}Tb - \frac{1}{2}T_{\rho\sigma}T^{\rho\sigma} - (T_{\mu\rho}n^{\mu})(T_{\nu}^{\rho}n^{\nu}) \geq 0 \quad (10)$$

and

$$\frac{1}{3}Ta - (T_{\mu\rho}n^\mu)(T_\nu^\rho n^\nu) \geq 0 \quad (11)$$

where $a = T_{\mu\nu}n^\mu n^\nu$ and $b = (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)n^\mu n^\nu$ are positive due to (9).

Also using the symmetry properties of $E_{\mu\nu}$, it is possible to decompose it with respect to any time like observer \vec{u} ($u^\alpha u_\alpha = -1$) as [5]

$$E_{\mu\nu} = -\left(\frac{\kappa_5}{\kappa_4}\right)^4 \left[\left(u_\mu u_\nu + \frac{1}{3}h_{\mu\nu} \right) U + 2(u_\mu Q_\nu) + P_{\mu\nu} \right]$$

with the properties

$$Q_\mu u^\mu = 0, \quad P_{(\mu\nu)} = P_{\mu\nu}, \quad P_\mu^\mu = 0, \quad P_{\mu\nu}u^\nu = 0$$

If we consider the Bianchi models then due to the symmetry of the spatial geometry we may choose

$$Q_\mu = P_{\mu\nu} = 0,$$

and the scalar part namely U is termed as dark energy density as it has energy-momentum tensor that of a radiation perfect fluid. So the restriction (8) implies that dark energy density should be always positive i.e.,

$$U \geq 0 \quad (12)$$

As it is not possible to make any restriction on $T_{\mu\nu}$ to satisfy inequations (10) and (11), so let us examine with some realistic model for the matter field.

III. EXAMPLES

(a) Perfect fluid model:

In this case the energy-momentum tensor has the form

$$T_{\mu\nu} = (\rho + p)n_\mu n_\nu + pg_{\mu\nu}, \quad n_\mu n^\mu = -1$$

with ρ and p as the energy density and isotropic pressure respectively.

The weak and strong energy conditions demand

$$a = \rho \geq 0 \quad \text{and} \quad 2b = \rho + 3p \geq 0 \quad (13)$$

Hence the inequations (7) (i.e., inequations (10) and (11)) take the form

$$\rho^2 \geq 0 \quad \text{and} \quad \rho(3p + 2\rho) \geq 0 \quad (14)$$

which are always true. Thus for perfect fluid model CNHC is automatically satisfied in brane scenarios if it is valid in general relativity.

(b) General form of energy-momentum tensor:

The general form of the brane energy momentum tensor for any matter fields (scalar field, perfect fluids, kinematic gases, dissipative fluids, etc.) including a combination of different fields can be covariantly written as [1]

$$T_{\mu\nu} = \rho n_\mu n_\nu + p h_{\mu\nu} + \Pi_{\mu\nu} + q_\mu n_\nu + q_\nu n_\mu \quad (15)$$

Here the energy flux q_μ and the anisotropic stress $\Pi_{\mu\nu}$ are projected, symmetric and traceless that is

$$q_\mu n^\mu = 0, \quad \Pi_{\mu\nu} n^\mu = 0, \quad \Pi_{\mu\nu} = \Pi_{\nu\mu}, \quad \Pi_{\mu\nu} g^{\mu\nu} = 0$$

Thus for this form of energy-momentum tensor, the restrictions on $S_{\mu\nu}$ now result

$$\frac{1}{3}\rho^2 - \frac{1}{2}\Pi_{\mu\nu}\Pi^{\mu\nu} \geq 0 \quad (16)$$

and

$$\frac{1}{3}\rho(3p + 2\rho) - q_\mu q^\mu \geq 0 \quad (17)$$

As for realistic matter, the energy density should be larger than the anisotropic stress and heat flux is very small in magnitude so the inequalities (16) and (17) are automatically satisfied. Hence the CNHC is satisfied for the above form of general energy-momentum tensor.

For future work, it will be interesting to find any general restrictions on $T_{\mu\nu}$ so that CNHC is automatically satisfied in brane world scenario.

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